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**Approximations to the
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Function of the Magnitude-
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Albert H. Nuttall
Surface Ship Sonar Department

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Preface

This research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, *Applications of Statistical Communication Theory to Acoustic Signal Processing*, Principal Investigator Dr. A. H. Nuttall (Code 33), Program Manager J. H. Probus, Naval Material Command (MAT 08T1); and under Project No. A12611, *Interarray Processing*, Principal Investigator, Dr. G. C. Carter (Code 313), Program Manager, J. Neely, Naval Sea Systems Command (SEA 63D).

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inversion of a nonlinearity that requires computer aid.

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Approximations to the Cumulative Distribution Function of The Magnitude-Squared Coherence Estimate

Introduction*

The cumulative distribution function of the estimate of magnitude-squared coherence, obtained by averaging over N statistically independent pieces of Gaussian data, is available in reference 1 as a sum of $N-1$ hypergeometric functions. Direct evaluation of this quantity has recently been simplified in reference 2; however, it is still a tedious and time-consuming calculation for large N and suffers from overflow and underflow unless special care is taken in programming. Furthermore, no simple result for obtaining confidence limits is available.

In this report, we investigate the suggestion of Fisher (reference 3) that the nonlinearity $\text{arc tanh}(\sqrt{x})$ converts the magnitude-squared coherence estimate to a near-Gaussian random variable. In particular, we present simple approximations for the mean and variance of this nonlinearly distorted magnitude-squared coherence estimate and fit a Gaussian cumulative distribution function over a wide range of: N , the number of pieces averaged in the magnitude-squared coherence estimate; C , the true magnitude-squared coherence; and P , the values of the cumulative distribution function. Inversion of the Gaussian cumulative distribution function affords a simple way of getting confidence limits for specified probabilities of threshold crossings.

Since the $\text{arc tanh}(\sqrt{x})$ nonlinear distortion takes no account of the known number, N , of pieces entering the magnitude-squared coherence estimate, an improved nonlinear distortion that utilizes this information is presented; it converts the magnitude-squared coherence estimates to more nearly a Gaussian random variable over a wider range of parameters N , C , and P . Evaluation of confidence limits requires the inversion of this nonlinearity; analytic inversion is not possible, but a numerical procedure converges rapidly.

Investigation of $\text{Arc tanh}(\sqrt{x})$ Nonlinearity

Moments

The probability density function of the magnitude-squared coherence estimate, \hat{C} , as given in reference 1, is

$$p_1(x) = (N-1) \frac{(1-C)^N (1-x)^{N-2}}{(1-Cx)^{2N-1}} F(1-N, 1-N; 1; Cx) \text{ for } 0 < x < 1 \text{ and } N \geq 2, \quad (1)$$

where N is the number of pieces averaged; C is the true magnitude-squared coherence; and F is a Gaussian hypergeometric function.

* Portions of this work, which were done jointly with G. C. Carter (NUSC), are being prepared for journal publication.

The nonlinearly distorted version of random variable \hat{C} , which we are interested in, is

$$D = \text{arc tanh}(\sqrt{\hat{C}}) = \frac{1}{2} \ln \left(\frac{1 + \sqrt{\hat{C}}}{1 - \sqrt{\hat{C}}} \right). \quad (2)$$

If the nonlinear operation in (2) results in a Gaussian random variable for D , then we will be interested in the mean and variance of D . This problem is considered analytically in appendix A; the only closed-form results that we can obtain are listed below. Let the m -th moment of D be denoted by

$$\mu_m(N, C) = \overline{D^m} = \int_0^1 dx [\text{arc tanh}(\sqrt{x})]^m p_1(x), \quad (3)$$

where the overbar denotes a statistical average. Then for $C = 0$, the mean of D is

$$\mu_1(N, 0) = \frac{\sqrt{\pi}}{2} \frac{\Gamma(N-1)}{\Gamma(N-1/2)} \sim \frac{\sqrt{\pi}/2}{\sqrt{N-1.25}} \quad \text{as } N \rightarrow \infty; \quad (4)$$

the m -th moment is

$$\mu_m(N, 0) \sim \frac{\Gamma(1 + m/2)}{(N-1)^{m/2}} \quad \text{as } N \rightarrow \infty; \quad (5)$$

and the variance is

$$\sigma^2(N, 0) = \mu_2(N, 0) - \mu_1^2(N, 0) \sim \frac{1-\pi/4}{N-1} \quad \text{as } N \rightarrow \infty. \quad (6)$$

This last expression is the asymptotic variance of D for $C = 0$.

In order to deduce the fundamental behavior of the mean and variance of D for $C \neq 0$, it was necessary to evaluate (3) numerically; this problem is considered in appendix B. The main results of the numerical investigation are listed below. We find mean

$$\mu_1(N, C) \cong \text{arc tanh}(\sqrt{C+B}) \quad \text{for } C > 0, \quad (7)$$

where

$$B = \frac{1-C^2}{2(N-1)}, \quad (8)$$

and variance

$$\sigma^2(N, C) \cong \frac{1}{2(N-1)} \quad \text{for } C > 0. \quad (9)$$

The imprecise qualifier $C > 0$ reflects the fact that the probabilistic behavior of \hat{C} and D is distinctly different for $C = 0$ versus $C > 0$; for example, compare (6) and (9). The precise region where (7), (8), and (9) are valid will become clear in later

plots. The result (9) is a slight modification of Fisher that better fits the calculated values of variance in appendix B and the slope of the cumulative distribution function plots. Its independence of true magnitude-squared coherence value C is striking and convenient.

Probabilistic Statements

If D is nearly a Gaussian random variable with mean $\mu_1(N, C)$ and variance $\sigma^2(N, C)$, then the cumulative distribution function of D , i.e., the probability that D is less than some threshold T , is given by the approximation

$$P_2(T) \equiv \text{Prob}\{D < T\} \cong \int_{-\infty}^T \frac{dx}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma^2}\right) = \Phi\left(\frac{T-\mu_1}{\sigma}\right), \quad (10)$$

where

$$\Phi(y) = \int_{-\infty}^y \frac{du}{\sqrt{2\pi}} \exp(-u^2/2) \quad (11)$$

is the cumulative distribution function for a zero-mean unit-variance Gaussian random variable. A plot of (10) on normal probability paper, with T as the linear abscissa, yields a straight line with abscissa-intercept μ_1 and slope $1/\sigma$, since the nonlinear transformation to plot ordinates in this case is the inverse function* $\Phi^{-1}(\cdot)$.

The exact cumulative distribution function of D is given by

$$\begin{aligned} P_2(T) &= \text{Prob}\{D < T\} = \text{Prob}\{\text{arc tanh } \sqrt{\hat{C}} < T\} \\ &= \text{Prob}\{\hat{C} < \tanh^2(T)\} = P_1(\tanh^2(T)), \end{aligned} \quad (12)$$

where we employed (1) and (2) and defined P_1 as the cumulative distribution function of \hat{C} :

$$P_1(A) = \text{Prob}\{\hat{C} < A\} = \int_0^A dx p_1(x). \quad (13)$$

Thus, we can relate cumulative distribution function P_2 to the known cumulative distribution function P_1 given in reference 1. Programs for the exact evaluation of (13) and (12) are given in appendices C and D, respectively.

The availability of approximation (10) for the cumulative distribution function of D enables us to give a simple approximation to the cumulative distribution function P_1 of \hat{C} ; namely

$$\begin{aligned} P_1(A) &= \text{Prob}\{\hat{C} < A\} = \text{Prob}\{D < \text{arc tanh } (\sqrt{A})\} \\ &= P_2(\text{arc tanh } (\sqrt{A})) \cong \Phi\left(\frac{\text{arc tanh } \sqrt{A} - \mu_1}{\sigma}\right), \end{aligned} \quad (14)$$

where μ_1 and σ are given by (7), (8), and (9).

*A superscript $I(\cdot)$ will be used to denote the inverse of a function.

Furthermore, we can now solve the inverse problem of determining a threshold A for specified values of cumulative distribution function P_1 and for given parameter values C and N . From (14), we have

$$A \cong \tanh^2(\mu_1 + \sigma \Phi^1(P_1)) , \quad (15)$$

where μ_1 and σ are in (7), (8), and (9). Simplification of (15) results in

$$A \cong \left(\frac{\alpha + \beta}{1 + \alpha\beta} \right)^2 , \quad (16)$$

where

$$\alpha = \left[C + \frac{1 - C^2}{2(N-1)} \right]^{1/2} , \quad (17)$$

$$\beta = \tanh \left[\frac{\Phi^1(P_1)}{\sqrt{2(N-1)}} \right] . \quad (18)$$

A program for the evaluation of (16), (17), and (18) is given in appendix E.

Plots

Before embarking on the plots of exact cumulative distribution function P_2 of distorted random variable D , as given by (12), we plot the exact cumulative distribution function P_1 in (13) of the original random variable \hat{C} , as given by reference 1. Plots on normal probability paper for $N = 8, 16, 32, 64, 128$ are given in figures 1-5*, respectively. In these figures, each dotted straight line corresponds to a Gaussian random variable with mean and variance as derived for magnitude-squared coherence estimate \hat{C} in reference 4, page 20. The discrepancy with the exact cumulative distribution function (in solid lines) indicates that \hat{C} is not well-approximated by a Gaussian random variable, especially for the small and large values of C and the extreme values of probability near .01 and .99.

Plots of the exact cumulative distribution function P_2 of D , given by (12), are presented in figures 6-10 for $N = 8, 16, 32, 64, 128$. Dotted straight lines corresponding to a Gaussian random variable satisfying (7)-(10) have been drawn for every C and N value being considered; however, they have been overdrawn by the exact cumulative distribution function (12) in some cases and are not visible. The agreement between exact and Gaussian cumulative distribution functions is extremely good except for very low values of C and N . Notice also that the curves are approximately parallel straight lines over a wide range and can, therefore, be interpolated more easily. Thus, approximate probabilistic relation (10) and its inverse (16) are very useful and accurate for a wide range of C and N , encompassing most of the useful values of these parameters.

* All figures are grouped at the end of the main text.

An Improved Nonlinear Transformation

Desired Transformation

The arc tanh (\sqrt{x}) nonlinear transformation in (2) is independent of the known quantity N , the number of pieces used in getting magnitude-squared coherence estimate \hat{C} . An improved transformation that utilizes N should, therefore, be possible and should be investigated to see if a more-nearly Gaussian random variable can result.

The method to determine this nonlinear transformation is as follows: suppose a random variable with cumulative distribution function $P_1(X)$ is given, and a nonlinear monotonically increasing transformation,

$$y = g(x) , \quad (19)$$

that will result in a specified cumulative distribution function $P_2(Y)$ is desired. We have

$$\begin{aligned} P_1(X) &= \text{Prob}\{x < X\} = \text{Prob}\{g(x) < g(X)\} \\ &= \text{Prob}\{y < g(X)\} = P_2(g(X)) . \end{aligned} \quad (20)$$

Therefore, inverting this equation, the desired nonlinearity is

$$g(x) = P_2^{-1}(P_1(x)) . \quad (21)$$

Thus, we need to know the inverse of the desired cumulative distribution function P_2 .

In particular, if y is Gaussian with mean μ_1 and variance σ^2 , then

$$\begin{aligned} P_2(Y) &= \Phi\left(\frac{Y - \mu_1}{\sigma}\right) , \\ P_2^{-1}(t) &= \mu_1 + \sigma\Phi^{-1}(t) , \end{aligned} \quad (22)$$

yielding

$$g(x) = \mu_1 + \sigma\Phi^{-1}(P_1(x)) . \quad (23)$$

This nonlinear transformation (23) will result in a Gaussian random variable with mean μ_1 and σ^2 . However, the quantities μ_1 and P_1 depend on true coherence C as well as N ; see (7) and (1). Since C is unknown (we are using \hat{C} to estimate C), (23) is useless unless it turns out that (23) is substantially independent of C ; it can, of course, depend on N . To ascertain this possibility, we have plotted, in figures 11, 12, and 13, the nonlinear distortion (23) for $N = 8, 16$, and 32 , respectively; the program is given in appendix F. Within each figure corresponding to a fixed value of N , the five distortions (23) for $C = .01, .05, .1, .5$, and $.99$ have been plotted as

solid lines; in addition, the arc tanh (\sqrt{x}) nonlinearity is superimposed in dotted lines. Furthermore, (23) has only been plotted in the range of probabilities [.01, .99] for P_1 , since this is the range we are concentrating on.

The ideal situation would occur if, within each figure, the same identical nonlinear function occurred for all five cases of $C = .01, .05, .1, .5$, and $.99$; of course, the function can change with N . Although this fortuitous situation does not occur precisely, inspection of each of the five cases within each figure reveals a marked independence of C . Thus, the desired goal of converting magnitude-squared coherence estimate \hat{C} to a nearly Gaussian random variable, by means of a nonlinear transformation that depends only on N , is achievable.

Now, the problem remains to find as simple an analytic transformation as possible to approximate the solid curves in figures 11, 12, and 13. A great deal of trial-and-error has resulted in the following candidate. The distorted version of \hat{C} we adopt is not (2), but rather

$$D_N = \text{arc tanh} (g_N(\hat{C})) , \quad (24)$$

where

$$g_N(x) = \sqrt{x} + \frac{(1-x)^N \ln \left(\frac{N-1}{2} x \right)}{2(N+2)} ; g_\infty(x) = \sqrt{x} . \quad (25)$$

The nonlinear transformation in (24) and (25) depends on N as well as x , but is independent of the unknown parameter C . The leading term of (25) is, of course, the earlier result. The correction term goes quickly to zero as $x \rightarrow 1$, and as N increases, as figures 11, 12, and 13 indicated desirable.

Probabilistic Statements

In order to measure the success of transformation (24) and (25), we need to evaluate the exact cumulative distribution function of random variable D_N . We have the exact relation

$$\begin{aligned} \text{Prob}\{D_N < T\} &= \text{Prob}\{\text{arc tanh}(g_N(\hat{C})) < T\} \\ &= \text{Prob}\{g_N(\hat{C}) < \tanh(T)\} \\ &= \text{Prob}\{\hat{C} < g_N^{-1}(\tanh(T))\} = P_1(g_N^{-1}(\tanh(T))) , \end{aligned} \quad (26)$$

where g_N^{-1} is the inverse function of g_N , and P_1 is the cumulative distribution function of \hat{C} . The program for evaluation of (26) is given in appendix G.

If we have succeeded in realizing a Gaussian random variable for D_N , then an approximation to (26) is given by

$$\text{Prob}\{D_N < T\} \cong \Phi \left(\frac{T - \mu_1}{\sigma} \right) , \quad (27)$$

where we have chosen to use (7), (8), and (9) here also. Therefore, an approximation to cumulative distribution function P_1 follows according to

$$\begin{aligned}
 P_1(A) &= \text{Prob}\{\hat{C} < A\} \\
 &= \text{Prob}\{\text{arc tanh}(g_N(\hat{C})) < \text{arc tanh}(g_N(A))\} \\
 &= \text{Prob}\{D_N < \text{arc tanh}(g_N(A))\} \\
 &\cong \Phi \left(\frac{\text{arc tanh}(g_N(A)) - \mu_1}{\sigma} \right). \quad (28)
 \end{aligned}$$

Furthermore, (28) affords an approximate expression for the required threshold value, A , to use for a specified probability value P_1 ; it is

$$A \cong g_N^1(\tanh(\mu_1 + \sigma \Phi^1(P_1))). \quad (29)$$

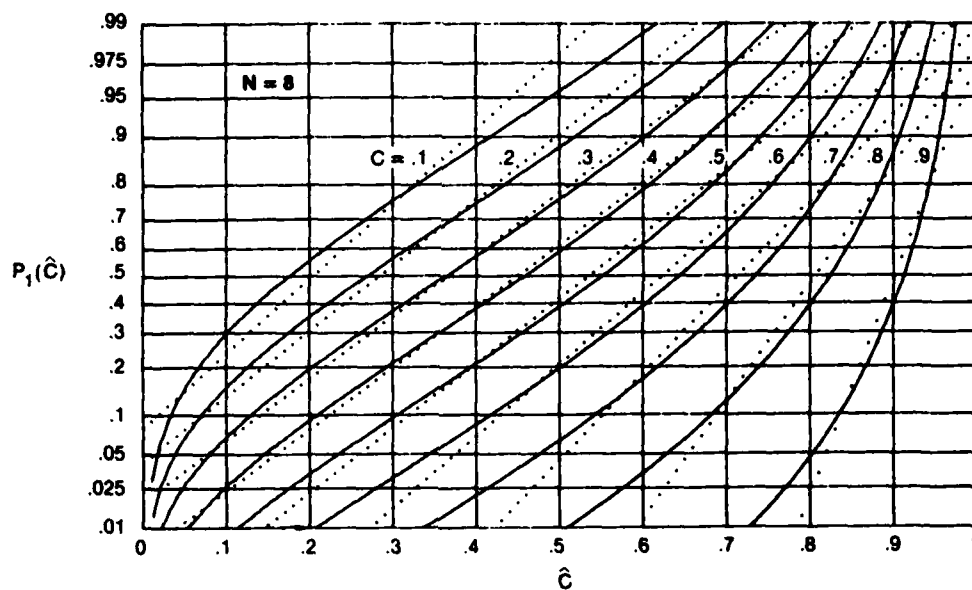
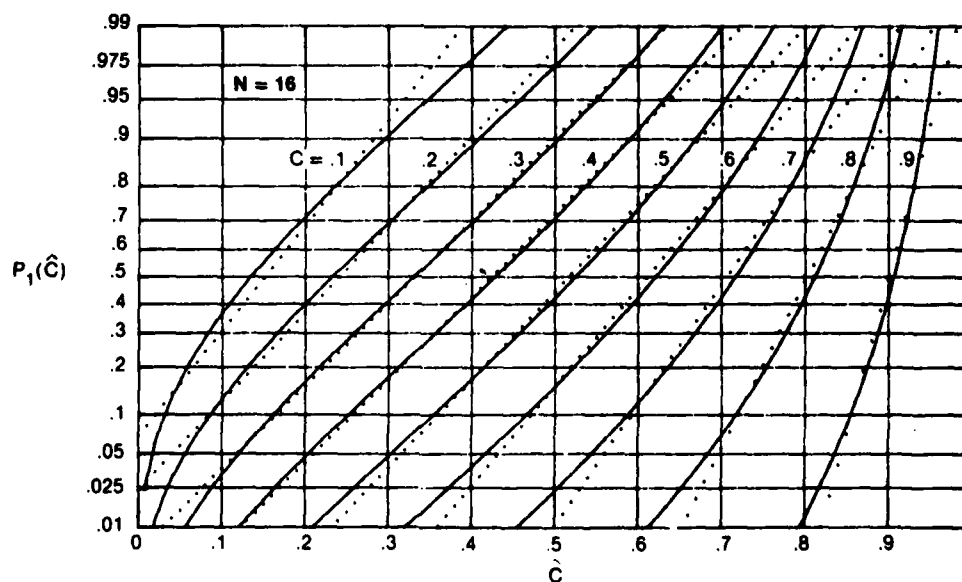
This relation generalizes (15), which applied to the arc tanh (\sqrt{x}) transformation. The main drawback to (26) and (29) is that we need the inverse function g_N^1 to (25). (This inverse is not needed for (27) or (28).) When the correction term in (25) is zero (as for $N = \infty$), the inverse relation is simply $g_\infty^1(y) = y^2$; however, no such simple relation exists for (25) in general, and g_N^1 must be found numerically. It would be worthwhile to uncover a simpler nonlinearity than (25) that still converts \hat{C} to a nearly Gaussian random variable, and, at the same time, is easily invertible; whether this is possible is unknown.

Plots

The final measure of success of transformation (24) and (25) is afforded by figures 14-18 for the cumulative distribution function of D_N for $N = 8, 16, 32, 64$, and 128, respectively. The discrepancy from a Gaussian random variable (dotted lines) is far less than for the corresponding cases in figures 6-10 for the arc tanh (\sqrt{x}) nonlinearity. Even for as low a value as $N = 8$, a fair approximation is given for $C = .01$.

Summary

Two nonlinear transformations have been presented that convert magnitude-squared coherence estimate \hat{C} to nearly a Gaussian random variable. They afford quick approximate evaluation of the cumulative distribution function of \hat{C} and can be used for arbitrarily large N ; see (14) and (28). They also allow for rapid calculation of confidence limits since these relations can be inverted to find the threshold required for a specified cumulative distribution function value; see (15) and (29). A drawback of the latter more-accurate result is that no simple analytical inversion to nonlinear transformation (25) exists; however, since computer aid would likely be used to evaluate the expressions given here, inversion of (25) is easily achieved. A simple recursive scheme was employed by the author; see appendix G.

Figure 1. Cumulative Distribution Function of \hat{C} for $N = 8$ Figure 2. Cumulative Distribution Function of \hat{C} for $N = 16$

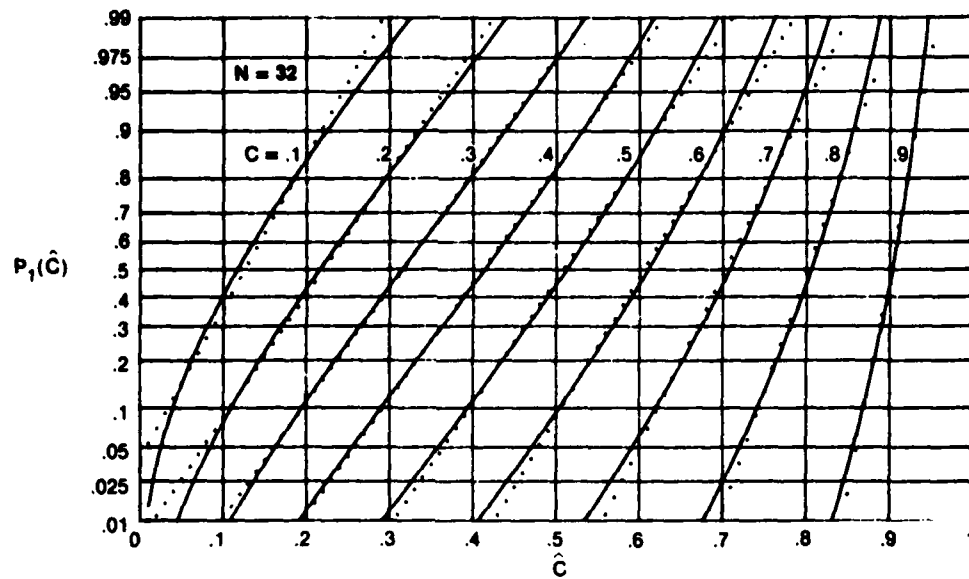


Figure 3. Cumulative Distribution Function of \hat{C} for $N = 32$

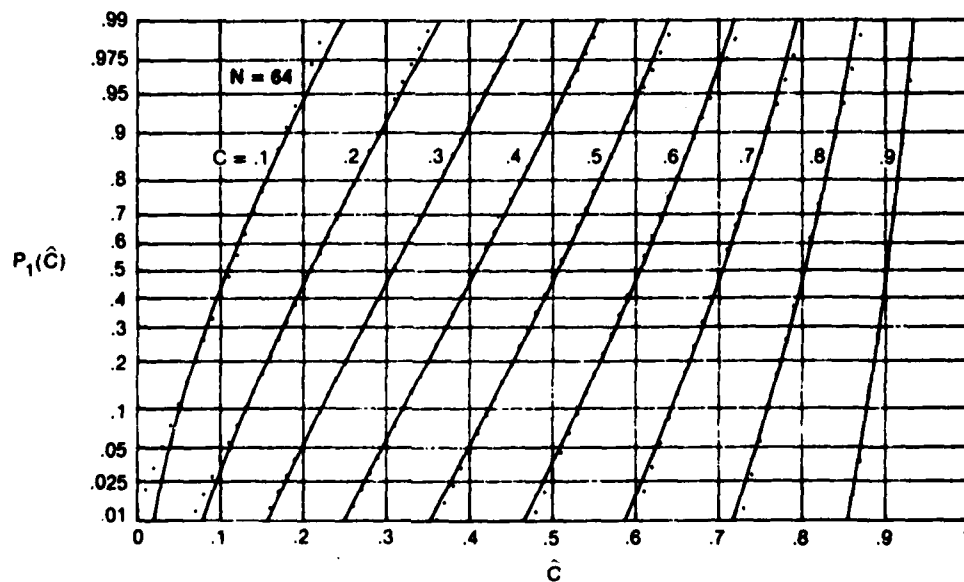
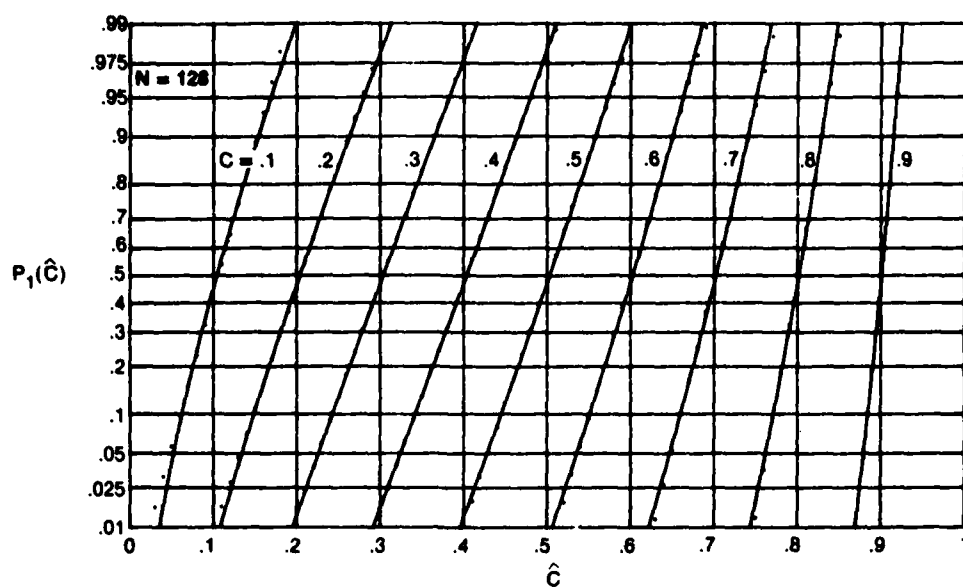
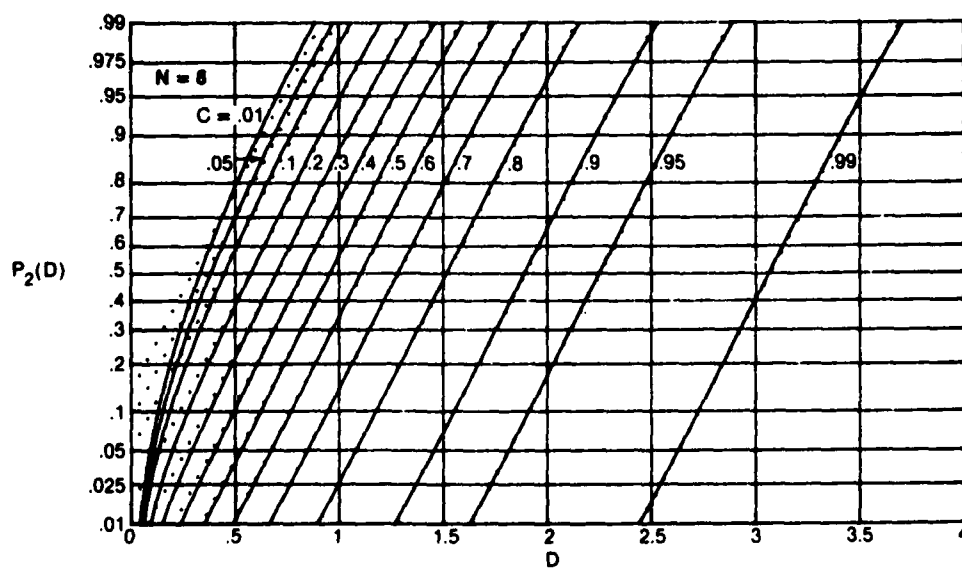


Figure 4. Cumulative Distribution Function of \hat{C} for $N = 64$

Figure 5. Cumulative Distribution Function of \hat{C} for $N = 128$ Figure 6. Cumulative Distribution Function of D for $N = 8$

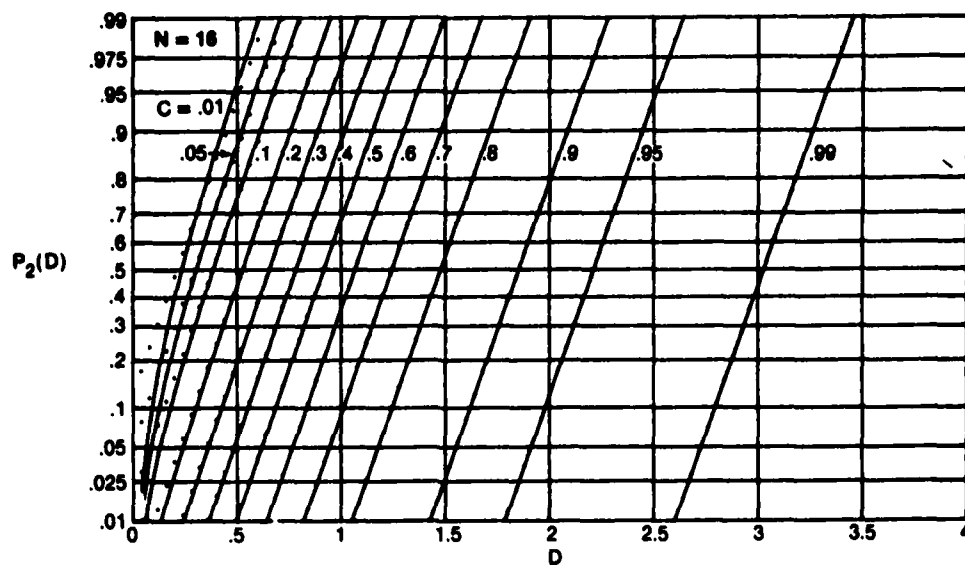


Figure 7. Cumulative Distribution Function of D for $N = 16$

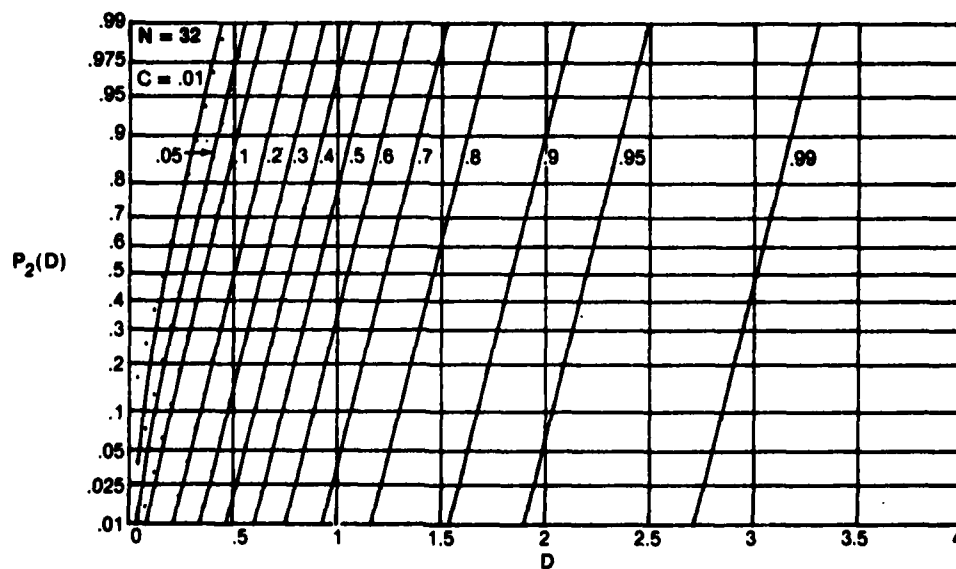


Figure 8. Cumulative Distribution Function of D for $N = 32$

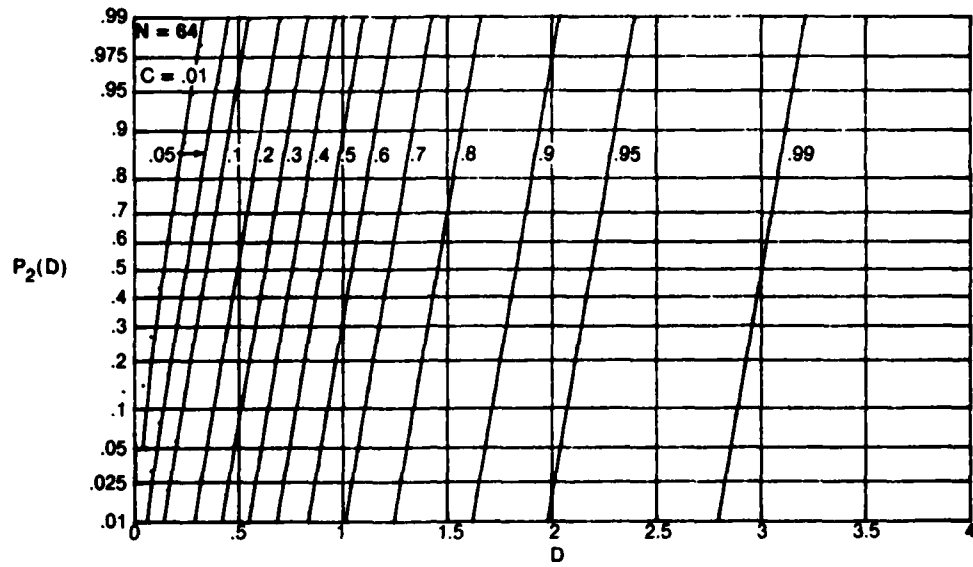


Figure 9. Cumulative Distribution Function of D for $N = 64$

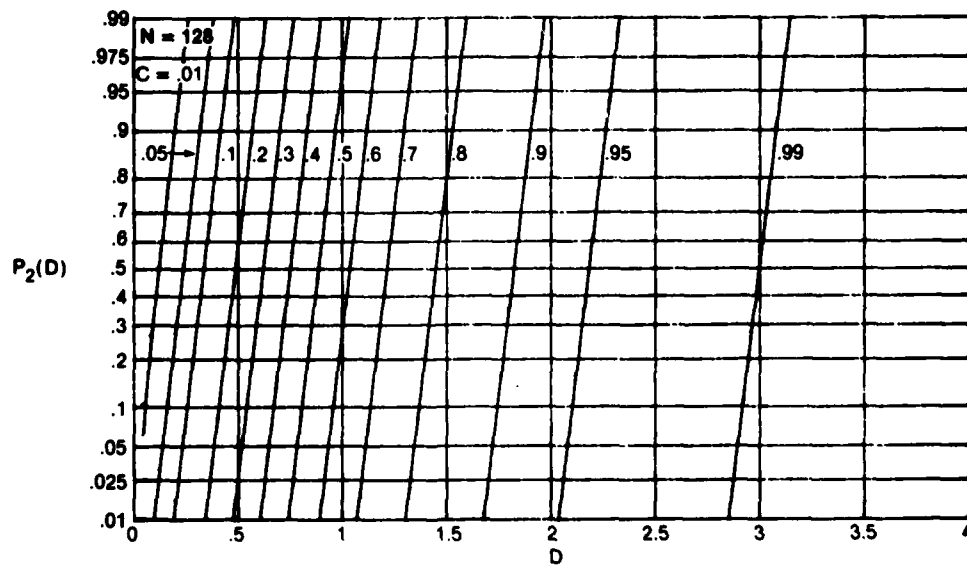
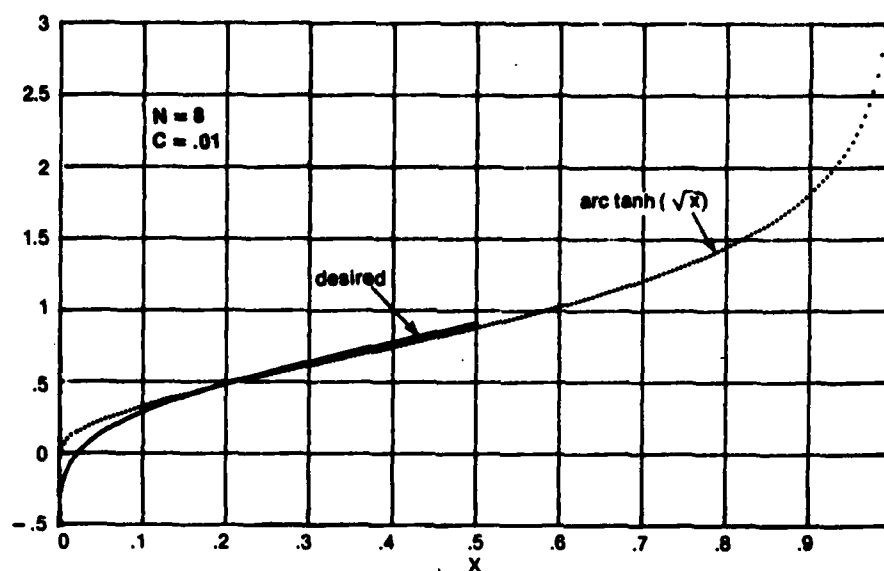
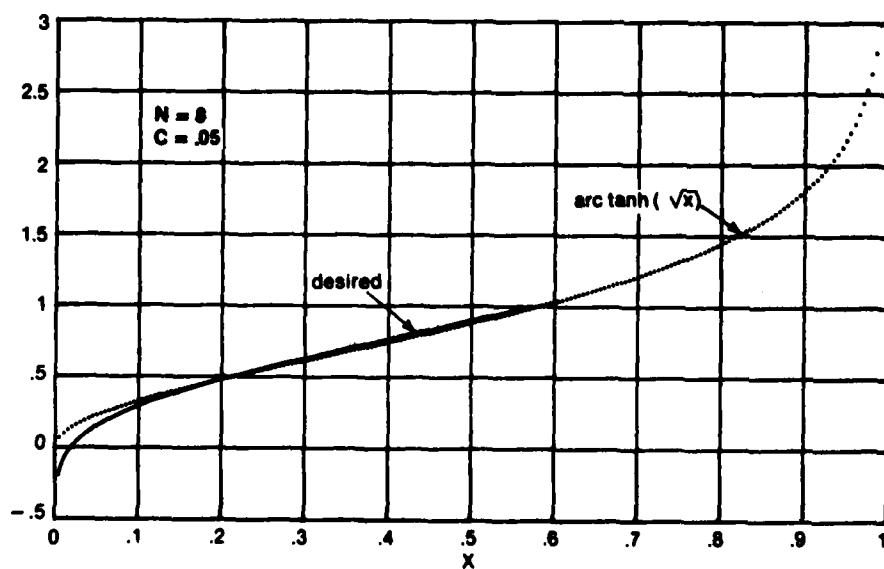


Figure 10. Cumulative Distribution Function of D for $N = 128$

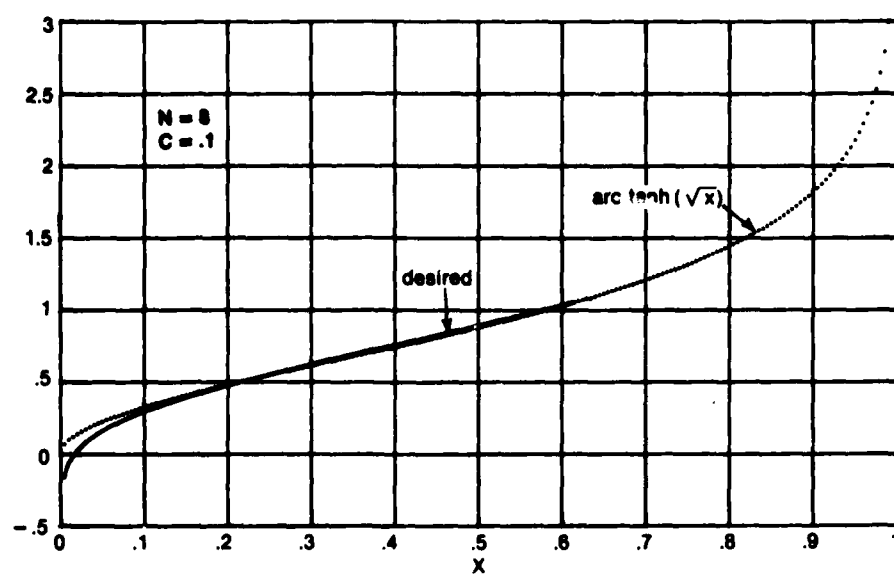
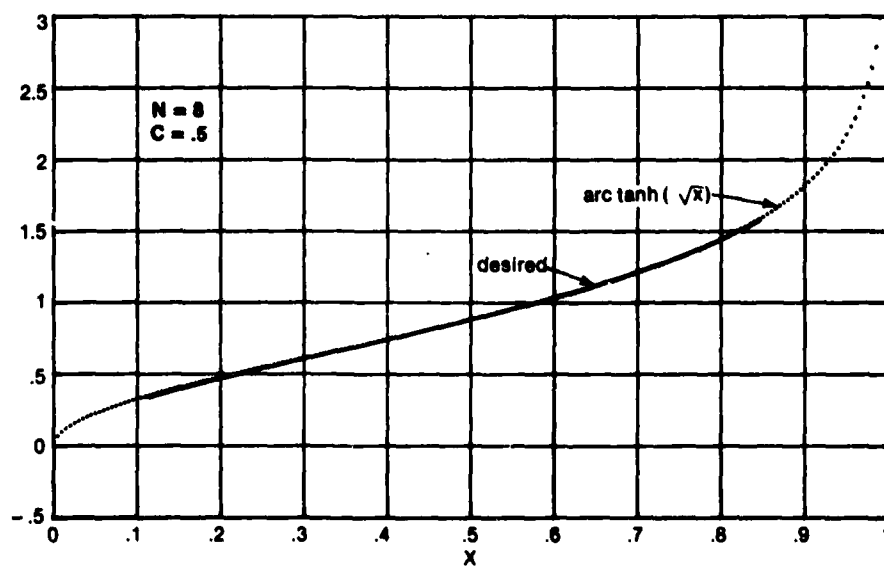


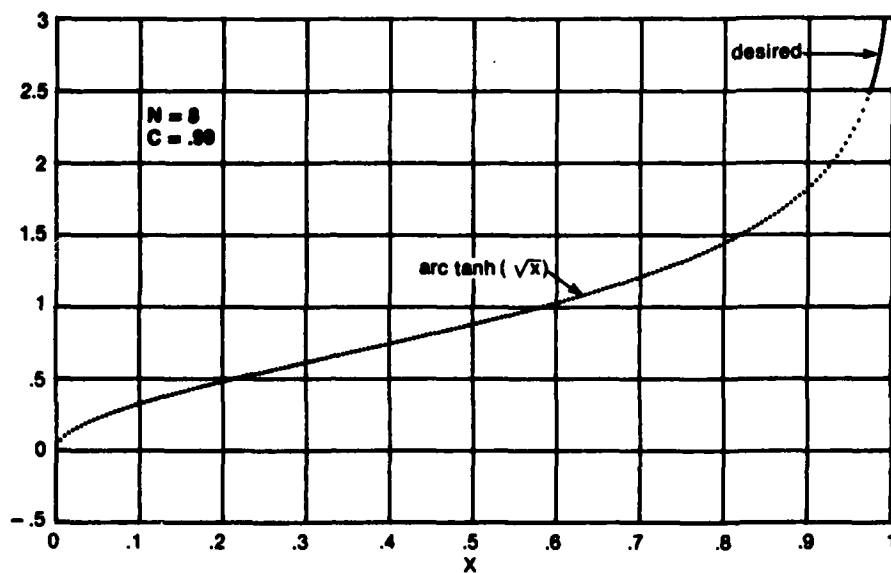
11a. $C = 0.01$



11b. $C = 0.05$

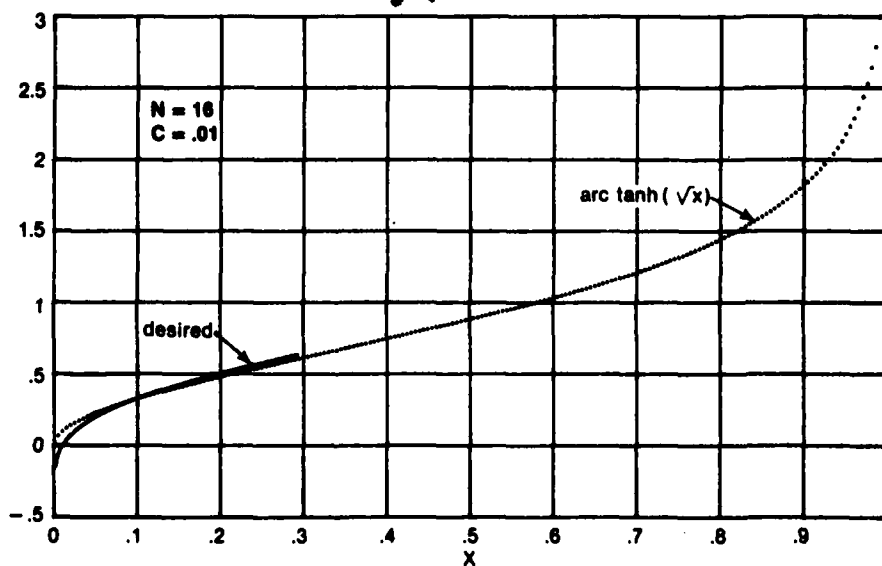
Figure 11. Desired Non-Linearity for $N = 8$

11c. $C = 0.1$ 11d. $C = 0.5$ Figure 11. (Cont'd) Desired Non-Linearity for $N = 8$



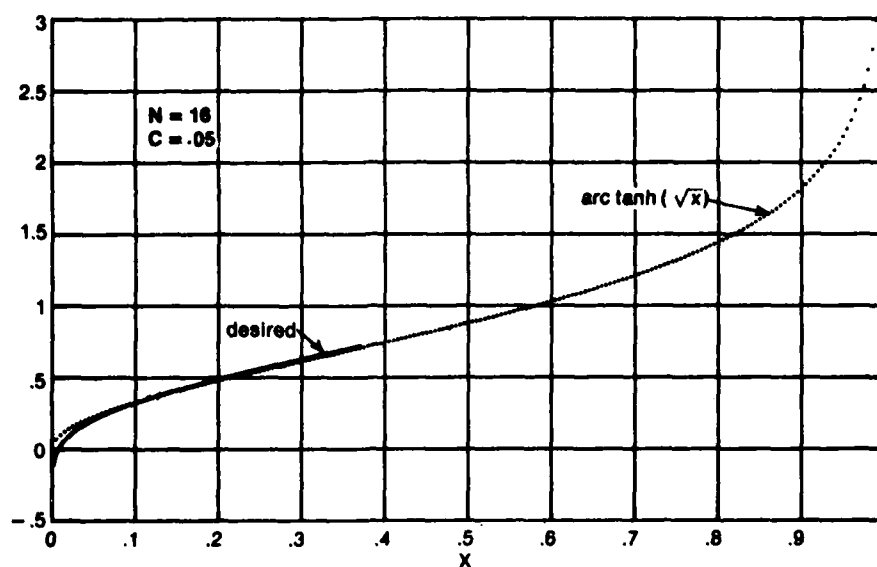
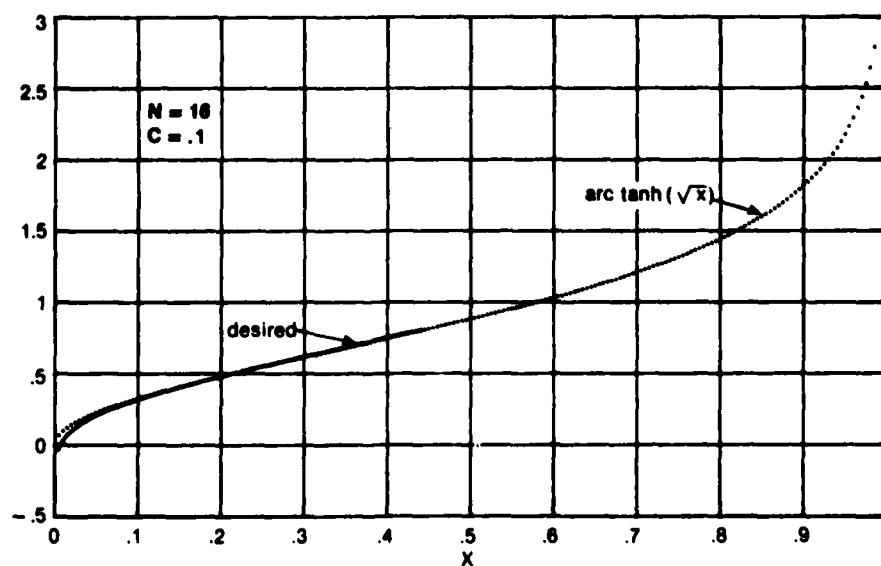
11e. $C = 0.99$

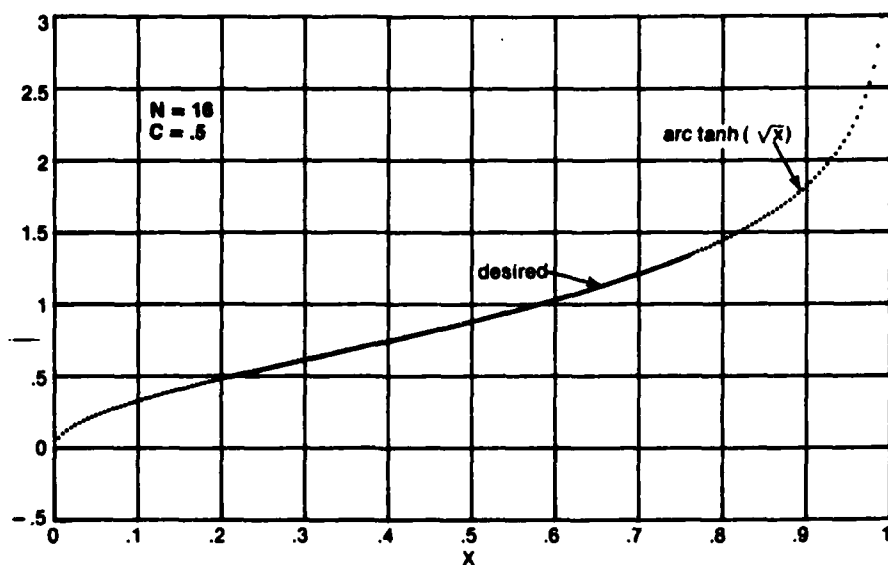
Figure 11. (Cont'd) Desired Non-Linearity for $N = 8$



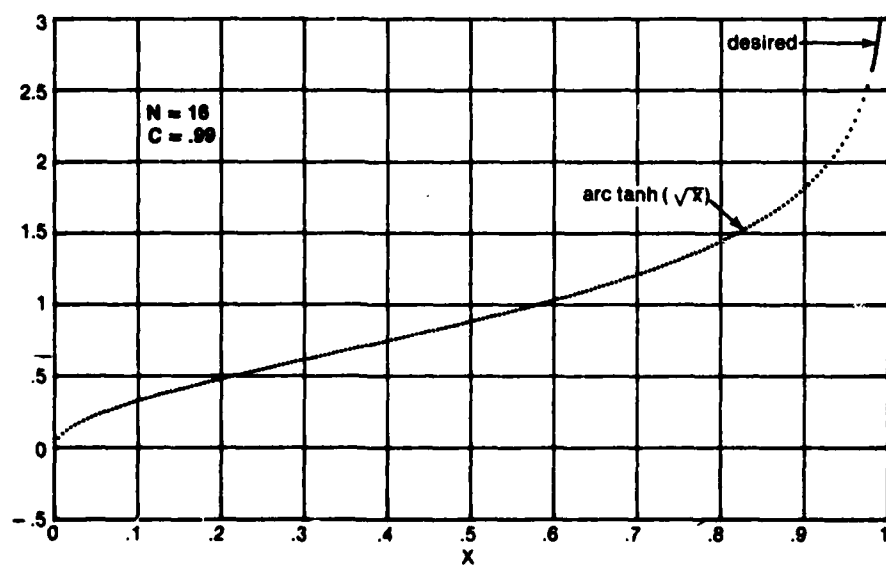
12a. $C = 0.01$

Figure 12. Desired Non-Linearity for $N = 16$

12b. $C = 0.05$ 12c. $C = 0.1$ Figure 12. (Cont'd) Desired Non-Linearity for $N = 16$

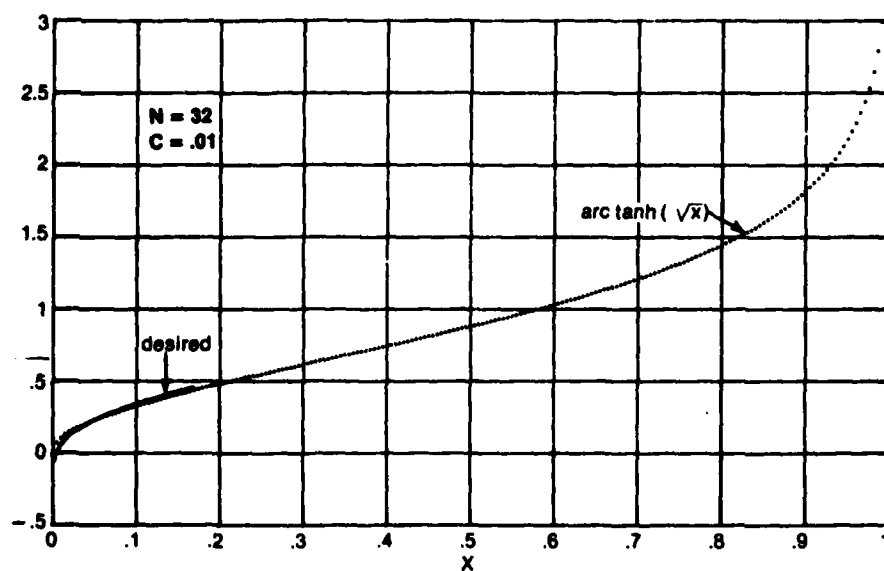
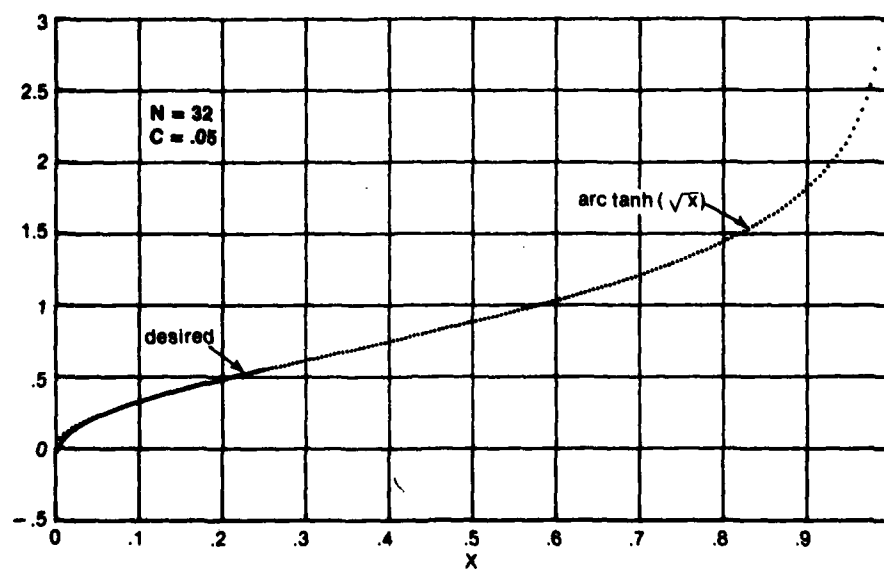


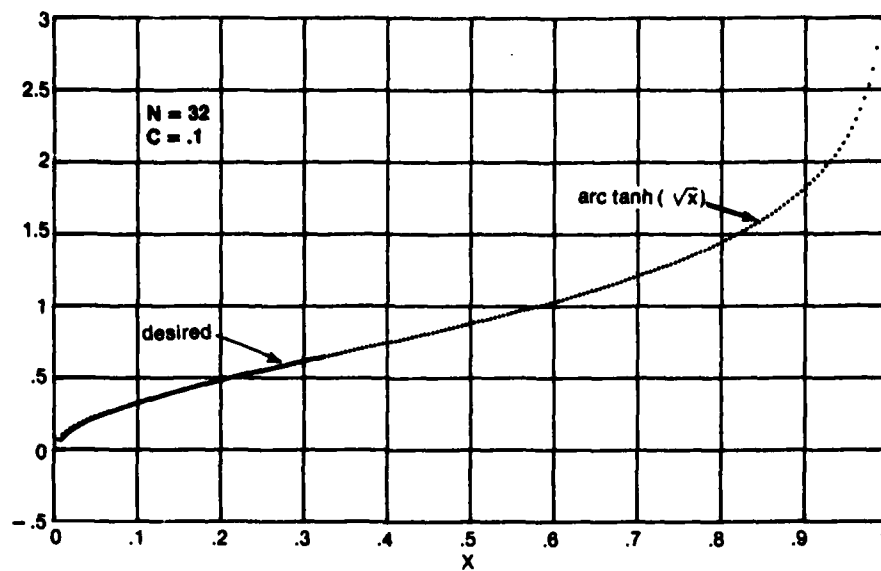
12d. $C = 0.5$



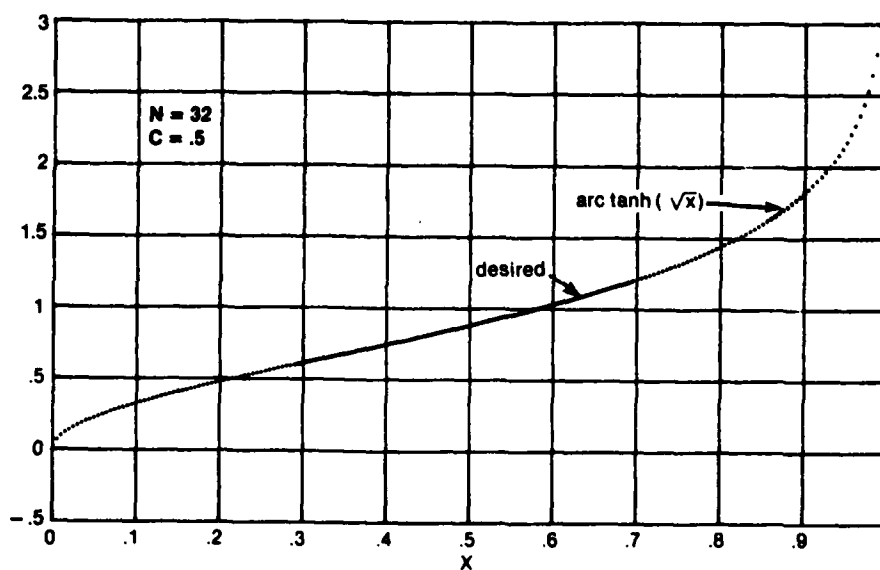
12e. $C = 0.99$

Figure 12. (Cont'd) Desired Non-Linearity for $N = 16$

13a. $C = 0.01$ 13b. $C = 0.05$ Figure 13. Desired Non-Linearity for $N = 32$

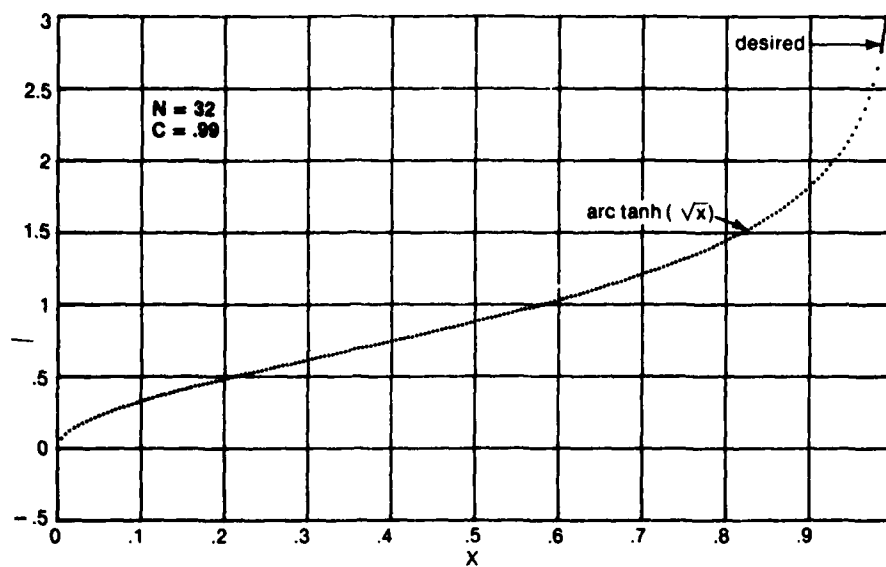
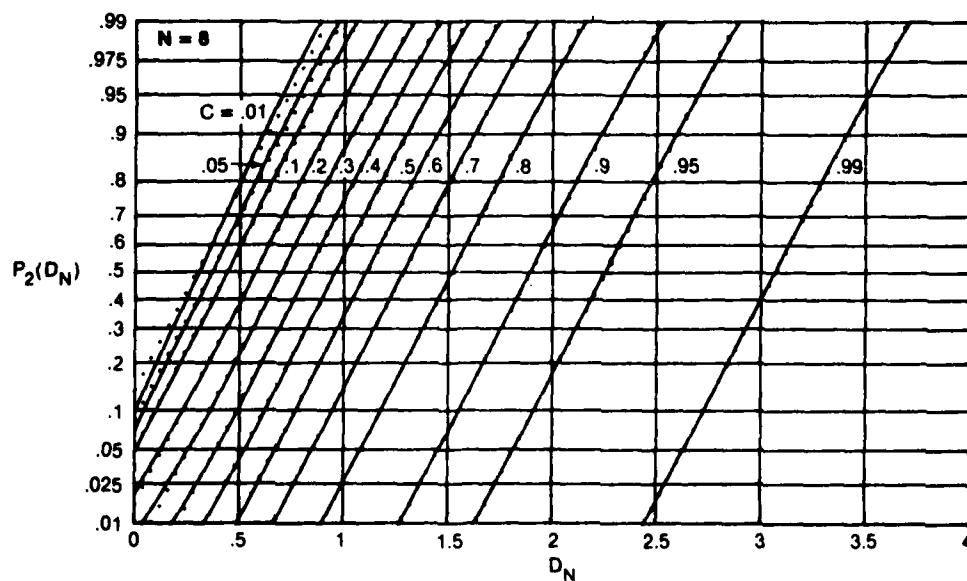


13c. $C = 0.1$



13d. $C = 0.5$

Figure 13. (Cont'd) Desired Non-Linearity for $N = 32$

13e. $C = 0.99$ Figure 13. (Cont'd) Desired Non-Linearity for $N = 32$ Figure 14. Cumulative Distribution Function of D_N for $N = 8$

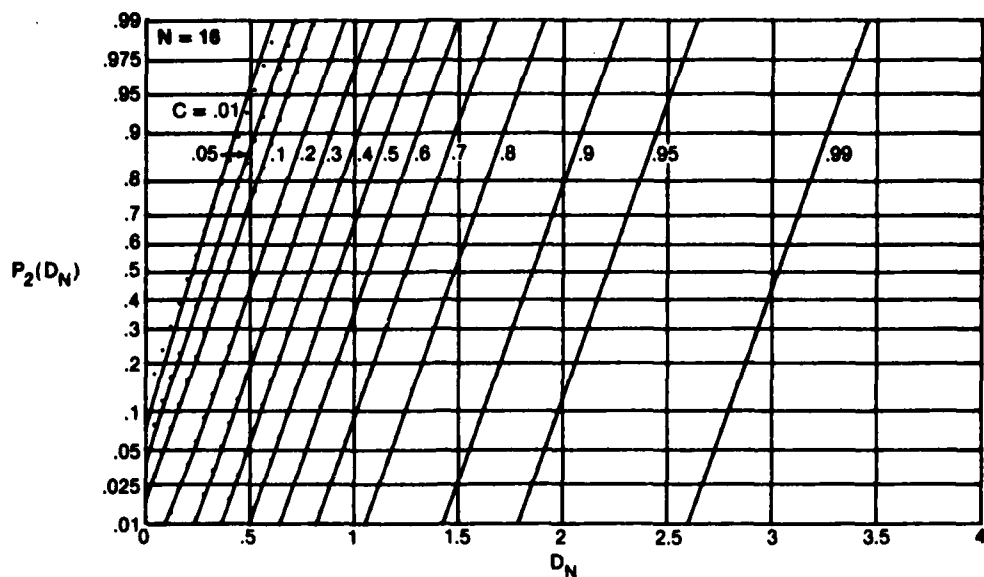


Figure 15. Cumulative Distribution Function of D_N for $N = 16$

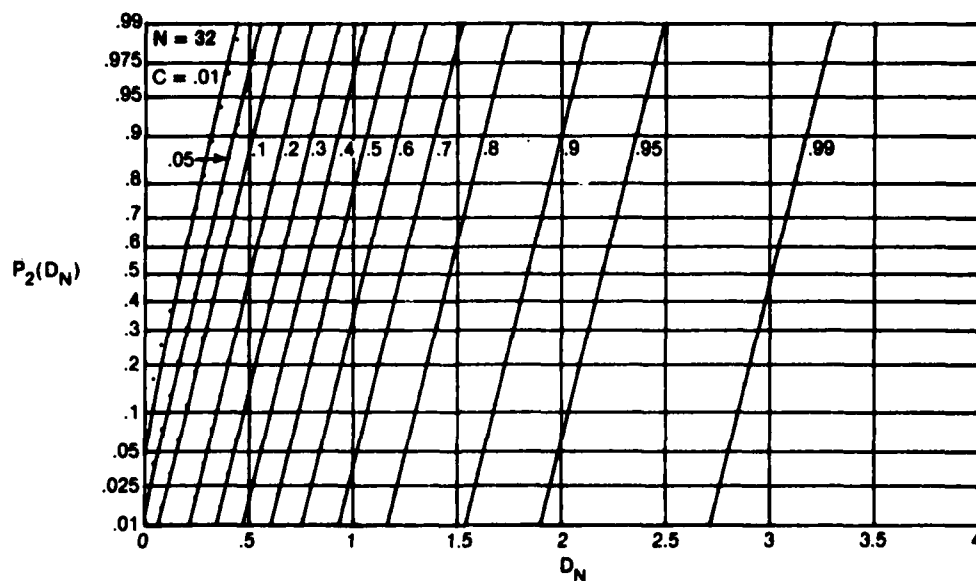
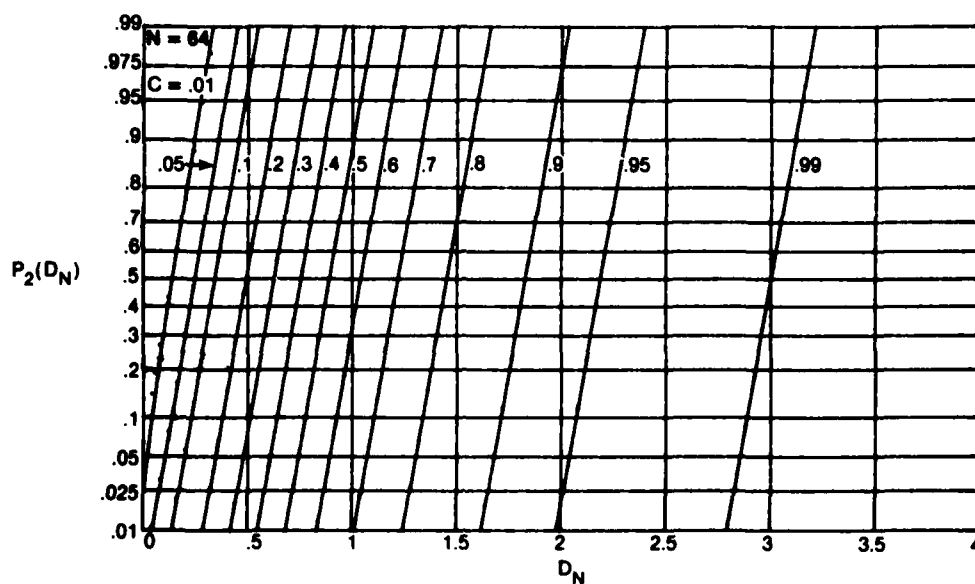
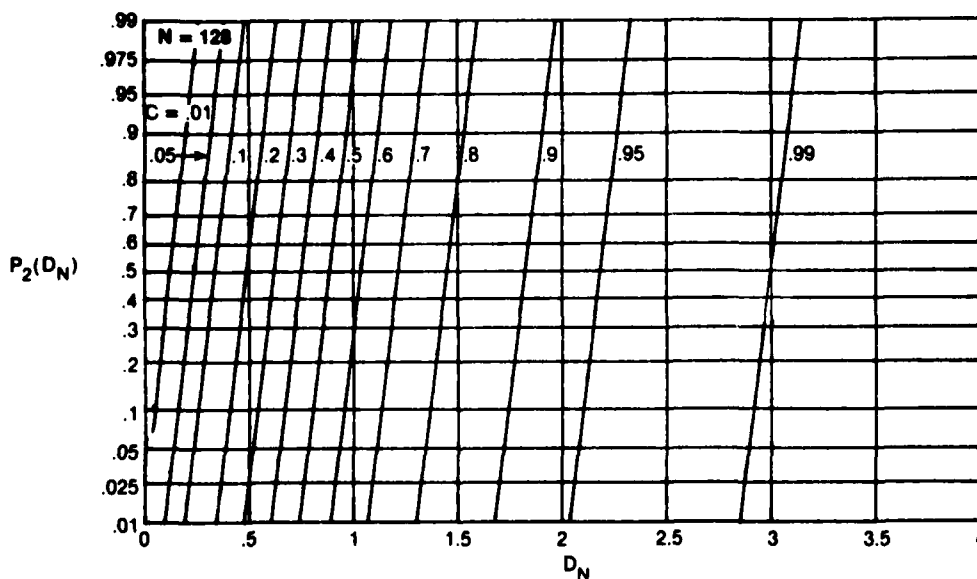


Figure 16. Cumulative Distribution Function of D_N for $N = 32$

Figure 17. Cumulative Distribution Function of D_N for $N = 64$ Figure 18. Cumulative Distribution Function of D_N for $N = 128$

Appendix A **Moments of D for C = 0**

Exact Results

For $C = 0$, the probability density function in (1), of magnitude-squared coherence estimate \hat{C} , simplifies to

$$p_1(x) = (N-1)(1-x)^{N-2} \text{ for } 0 < x < 1 \text{ and } N \geq 2. \quad (\text{A-1})$$

The m-th moment (3) is then given by

$$\mu_m(N, 0) = \int_0^1 dx p_1(x) \operatorname{arctanh}^m(\sqrt{x}). \quad (\text{A-2})$$

Let

$$x = \tanh^2(t), \quad dx = dt \, 2 \sinh(t) / \cosh^3(t), \quad (\text{A-3})$$

to obtain

$$\begin{aligned} \mu_m(N, 0) &= 2 \int_0^{\infty} dt \frac{\sinh(t)}{\cosh^3(t)} p_1(\tanh^2(t)) t^m \\ &= 2(N-1) \int_0^{\infty} dt \frac{\sinh(t)}{\cosh^{2N-1}(t)} t^m. \end{aligned} \quad (\text{A-4})$$

For $m = 0$, the substitution $u = \cosh(t)$ immediately yields unit area. So, let $m \geq 1$ in the following. Integrate by parts with

$$u = t^m, \quad dv = dt \frac{\sinh(t)}{\cosh^{2N-1}(t)}, \quad (\text{A-5})$$

and obtain

$$\mu_m(N, 0) = m \int_0^{\infty} dt \frac{t^{m-1}}{\cosh^{2N-2}(t)} \text{ for } m \geq 1. \quad (\text{A-6})$$

We now specialize to $m = 1$; let $x = e^t$, $dx = x dt$, $p = 2N-2$, and obtain

$$\begin{aligned} \mu_1(N, 0) &= 2^{p-1} \int_0^{\infty} dx \frac{x^{p-1}}{(1+x^2)^p} = 2^{2N-2} \frac{\Gamma^2(N-1)}{\Gamma(2N-2)} \\ &= \frac{\sqrt{\pi}}{2} \frac{\Gamma(N-1)}{\Gamma(N-\frac{1}{2})} = \mu_1(N-1, 0) \frac{N-2}{N-1.5} \text{ for } N \geq 3; \mu_1(2, 0) = 1, \end{aligned} \quad (\text{A-7})$$

where we employed reference 5, 3.241 4, and reference 6, 6.1.18. Then using reference 6, 6.1.47, with the parameters selected to eliminate the first correction term, we find

$$\mu_1(N,0) \sim \frac{\sqrt{\pi}/2}{\sqrt{N-1.25}} \quad \text{as } N \rightarrow \infty. \quad (\text{A-8})$$

Asymptotic Expansion to One Term

We were unable to locate a closed-form expression for (A-6) when $m \geq 2$. A recursion relation similar to (A-7) is derivable for $m \geq 2$ and will be presented later. But, a useful approximation for any m and for large N follows quickly from (A-6):

$$\begin{aligned} \frac{1}{\cosh(t)} &\sim \frac{1}{1+t^2/2} \sim \exp(-t^2/2) \quad \text{as } t \rightarrow 0; \\ \frac{1}{\cosh^p(t)} &\sim \exp(-pt^2/2) \quad \text{as } t \rightarrow 0. \end{aligned} \quad (\text{A-9})$$

Therefore,

$$\begin{aligned} \mu_m(N,0) &\sim m \int_0^\infty dt \, t^{m-1} \exp(-pt^2/2) \\ &= \frac{\Gamma(\frac{m}{2}+1)}{(N-1)^{m/2}} \quad \text{as } N \rightarrow \infty. \end{aligned} \quad (\text{A-10})$$

(The reason that (A-10) for $m = 1$ does not agree precisely with (A-8) is that (A-8) is exact to order $N^{-3/2}$, whereas (A-10) is correct only to order $N^{-1/2}$ for $m = 1$. This shortcoming is alleviated below.)

From (A-10), we can now evaluate the variance of D for $C = 0$; it is

$$\sigma^2 = \mu_2(N,0) - \mu_1^2(N,0) \sim \frac{1-\pi/4}{N-1} \quad \text{as } N \rightarrow \infty. \quad (\text{A-11})$$

Asymptotic Expansion to Two Terms

Our starting point is (A-6):

$$\mu_m(N,0) = m \int_0^\infty dx \, \frac{x^{m-1}}{\cosh^p(x)} \quad (\text{A-12})$$

Let

$$\cosh(x) = e^t, \quad t = \ln \cosh(x), \quad x = \operatorname{arc} \cosh(e^t),$$

$$\frac{dx}{dt} = \frac{e^t}{\sqrt{e^{2t} - 1}} \quad (\text{A-13})$$

when we used reference 6, 4.6.38. Then

$$\mu_m(N, 0) = m \int_0^\infty dt \, f_1(t) f_2^{m-1}(t) e^{-pt}, \quad (\text{A-14})$$

where

$$f_1(t) = \frac{e^t}{\sqrt{e^{2t} - 1}}, \quad f_2(t) = \operatorname{arc} \cosh(e^t), \quad p = 2N - 2. \quad (\text{A-15})$$

We are now in a position to employ Watson's Lemma, reference 7, pp. 102-103. We need the expansions of $f_1(t)$ and $f_2(t)$ about $t = 0$. Consider

$$\begin{aligned} f_1(t) &= \frac{1+t+O(t^2)}{\sqrt{2t+2t^2+O(t^3)}} = \frac{1}{\sqrt{2t}} \frac{1+t+O(t^2)}{1+\frac{1}{2}t+O(t^2)} \\ &= \frac{1}{\sqrt{2t}} \left[1 + \frac{1}{2}t + O(t^2) \right]. \end{aligned} \quad (\text{A-16})$$

Also via reference 6, 4.6.21 and 4.1.24,

$$\begin{aligned} f_2(t) &= \ln \left[e^t + \sqrt{e^{2t} - 1} \right] \\ &= \ln \left[1+t+O(t^2) + \sqrt{2t+2t^2+O(t^3)} \right] \\ &= \ln \left[1+t+O(t^2) + \sqrt{2t} \left\{ 1+\frac{1}{2}t+O(t^2) \right\} \right] \\ &= \ln \left[1+\sqrt{2t}+t+\frac{1}{\sqrt{2}}t^{3/2}+O(t^2) \right] \\ &= \sqrt{2t}+t+\frac{1}{\sqrt{2}}t^{3/2}+O(t^2) - \frac{1}{2} \left[\sqrt{2t}+t+O(t^{3/2}) \right]^2 + \frac{1}{3} \left[\sqrt{2t}+O(t) \right]^3 \\ &= \sqrt{2t}+t+\frac{1}{\sqrt{2}}t^{3/2}+O(t^2) - \frac{1}{2} \left[2t+2\sqrt{2}t^{3/2}+O(t^2) \right] + \frac{1}{3} \left[2\sqrt{2}t^{3/2}+O(t^2) \right] \\ &= \sqrt{2t} \left[1 + \frac{1}{6}t + O(t^{3/2}) \right]. \end{aligned} \quad (\text{A-17})$$

Therefore,

$$f_2^{m-1}(t) = (2t)^{\frac{m-1}{2}} \left[1 + \frac{m-1}{6} t + O(t^2) \right], \quad (\text{A-18})$$

and using (A-16)

$$f_1(t) f_2^{m-1}(t) = (2t)^{\frac{m-1}{2}} \left[1 + \frac{m+2}{6} t + O(t^2) \right]. \quad (\text{A-19})$$

Term-by-term integration of (A-19) in integral (A-14) yields the desired result

$$\begin{aligned} \mu_m(N, 0) &\sim \frac{\Gamma(\frac{m}{2} + 1)}{(N-1)^{m/2}} \left[1 + \frac{m(m+2)}{24(N-1)} \right] \\ &\sim \frac{\Gamma(\frac{m}{2} + 1)}{(N - \frac{1+m}{12})^{m/2}} \quad \text{as } N \rightarrow \infty. \end{aligned} \quad (\text{A-20})$$

Special cases are

$$\mu_1(N, 0) \sim \frac{\sqrt{\pi}/2}{(N-1)^{1/2}} \left[1 + \frac{1}{8(N-1)} \right] \sim \frac{\sqrt{\pi}/2}{\sqrt{N-1.25}} \quad \text{as } N \rightarrow \infty; \quad (\text{A-21})$$

$$\mu_2(N, 0) \sim \frac{1}{N-1} \left[1 + \frac{1}{3(N-1)} \right] \sim \frac{1}{N-4/3} \quad \text{as } N \rightarrow \infty; \quad (\text{A-22})$$

$$\begin{aligned} \sigma^2 = \mu_2 - \mu_1^2 &\sim \frac{1 - \pi/4}{N-1} + \left(\frac{1}{3} - \frac{\pi}{16} \right) \frac{1}{(N-1)^2} \\ &\sim \left(1 - \frac{\pi}{4} \right) \left(N - \frac{64 - 15\pi}{48 - 12\pi} \right)^{-1} \quad \text{as } N \rightarrow \infty. \end{aligned} \quad (\text{A-23})$$

Equation (A-21) agrees precisely with (A-8) when the latter is expanded to two terms in $(N-1)^{-1}$. More generally, (A-20) extends (A-10) to two terms in the asymptotic expansion, and (A-23) generalizes (A-11) to two terms. Comparison of (A-20)-(A-23) with results of the numerical procedure in appendix B reveals very close agreement.

Third Cumulant of Normalized D for $C = 0$

In order to determine whether random variable D , for $C = 0$, is approaching a Gaussian random variable as N increases, we evaluate the third cumulant of the normalized random variable

$$r = \frac{D - \mu_1}{\sigma} \quad (\text{A-24})$$

We have the usual first two cumulants of r :

$$\lambda_1^{(r)} = 0, \quad \lambda_2^{(r)} = 1. \quad (\text{A-25})$$

Then, using (A-20)-(A-23),

$$\begin{aligned} \lambda_3^{(r)} &= \overline{(r - \bar{r})^3} = \bar{r}^3 = \frac{\overline{(D - \mu_1)^3}}{\sigma^3} \\ &= \frac{\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3}{\sigma^3} \sim \frac{2\sqrt{\pi}(\pi-3)}{(4-\pi)^{3/2}} = .631 \quad \text{as } N \rightarrow \infty. \end{aligned} \quad (\text{A-26})$$

Since this quantity does not decrease to zero as N increases to ∞ , the nonlinear distortion D of magnitude-squared coherence estimate \hat{C} does not approach Gaussian, at least in the case when $C = 0$.

Recursion for $\mu_m(N, 0)$

From (A-1) and (A-2),

$$\mu_m(N, 0) = \int_0^1 dx (N-1)(1-x)^{N-2} A^m(\sqrt{x}), \quad (\text{A-27})$$

where

$$A(t) = \text{arctanh}(t), \quad A'(t) = \frac{1}{1-t^2}. \quad (\text{A-28})$$

Let $t = \sqrt{x}$ in (A-27) to obtain

$$\mu_m(N, 0) = \int_0^1 dt \, 2t(N-1)(1-t^2)^{N-2} A^m(t). \quad (\text{A-29})$$

Let $m \geq 1$ in the following and integrate by parts with $u = A^m(t)$, $v = -(1-t^2)^{N-1}$, to obtain

$$\mu_m(N, 0) = m \int_0^1 dt (1-t^2)^{N-2} A^{m-1}(t). \quad (\text{A-30})$$

Now we assume that $N \geq 3$ and develop

$$\begin{aligned}\mu_m(N, 0) &= m \int_0^1 dt (1-t^2)^{N-3} (1-t^2)^{m-1} A^{m-1}(t) \\ &= \mu_m(N-1, 0) - m \int_0^1 dt (1-t^2)^{N-3} t^2 A^{m-1}(t).\end{aligned}\quad (\text{A-31})$$

Integrate by parts with $u = tA^{m-1}(t)$, $v = -(1-t^2)^{N-2}/[2(N-2)]$, to obtain

$$\mu_m(N, 0) = \mu_m(N-1, 0) - \frac{1}{2(N-2)} \mu_m(N, 0) - \frac{m(m-1)}{2(N-2)} \int_0^1 dt (1-t^2)^{N-3} t A^{m-2}(t). \quad (\text{A-32})$$

For $m = 1$, this immediately reduces to the recursion in (A-7). Furthermore, for $m = 2$, the last integral can be evaluated immediately, and we find the convenient recursion for the second moment:

$$\mu_2(N, 0) = \frac{2N-4}{2N-3} \mu_2(N-1, 0) - \frac{1}{(N-2)(2N-3)} \text{ for } N \geq 3; \mu_2(2, 0) = 2 \ln 2. \quad (\text{A-33})$$

The starting value for $\mu_2(2, 0)$ follows from (A-30) and (A-28):

$$\mu_2(2, 0) = 2 \int_0^1 dt A(t) = 2 \int_0^1 dt \int_0^1 \frac{dx}{1-x^2} = 2 \int_0^1 \frac{dx}{1-x^2} \int_0^1 dt = 2 \int_0^1 \frac{dx}{1+x} = 2 \ln 2. \quad (\text{A-34})$$

We now assume that $m \geq 3$ and conclude the recursion derivation in (A-32). Integrate by parts with $u = A^{m-2}(t)$, $v = -(1-t^2)^{N-2}/[2(N-2)]$, to obtain the final result

$$\mu_m(N, 0) = \frac{2N-4}{2N-3} \mu_m(N-1, 0) - \frac{m(m-1)}{(2N-3)(2N-4)} \mu_{m-2}(N-1, 0) \text{ for } N \geq 3 \text{ and } m \geq 3. \quad (\text{A-35})$$

(Equation (A-35) actually holds for $m = 2$ also; in fact, it reduces to (A-33).)

In order to start this recursion, we need to know

$$\mu_m(2, 0) = m \int_0^1 dt A^{m-1}(t) = \frac{m}{2^{m-1}} \int_0^1 dt \left(\ln \frac{1+t}{1-t} \right)^{m-1}. \quad (\text{A-36})$$

We already know $\mu_2(2, 0)$ in (A-34), so keep $m \geq 3$ in the following. Let $x = (1-t)/(1+t)$ and obtain

$$\mu_m(2, 0) = \frac{m}{2^{m-2}} \int_0^1 \frac{dx}{(1+x)^2} (-\ln x)^{m-1}. \quad (\text{A-37})$$

Integrate by parts with $u = (-\ln x)^{m-1}$, $v = x/(1+x)$, to obtain

$$\mu_m(2,0) = \frac{m(m-1)}{2^{m-2}} \int_0^1 \frac{dx}{1+x} (-\ln x)^{m-2} \quad (\text{A-38})$$

Now let $x = e^{-t}$ and get

$$\mu_m(2,0) = \frac{m(m-1)}{2^{m-2}} \int_0^\infty \frac{dt}{e^t + 1} t^{m-2} \quad (\text{A-39})$$

Use reference 6, 23.2.8 and obtain

$$\mu_m(2,0) = m! \frac{2^{m-2} - 1}{2^{m-2}} \zeta(m-1) \text{ for } m \geq 3, \quad (\text{A-40})$$

where ζ is the Riemann zeta function. Thus, from reference 6, 23.2.24 and 23.2.25,

$$\mu_2(2,0) = \frac{\pi^2}{4}, \quad \mu_3(2,0) = \frac{7}{48} \pi^4. \quad (\text{A-41})$$

The value for $\mu_4(2,0)$ requires knowledge of $\zeta(3)$; values of $\zeta(n)$ for $n \geq 2$ are given in reference 6, page 811. Thus, we now have all the starting values necessary to use recursion (A-35) for $m \geq 3$.

Appendix B Numerical Evaluation of Moments of D

Our starting point is the top line of (A-4), modified to the case for $C \neq 0$. We then express the hyperbolic functions in terms of exponentials and obtain

$$\mu_m(N, C) = 8 \int_0^{\infty} dt E \frac{1-E}{(1+E)^3} p_1\left(\frac{(1-E)^2}{(1+E)}\right) t^m, \quad (B-1)$$

where

$$E = \exp(-2t), \quad (B-2)$$

and p_1 is given by (1). A program for evaluation of (B-1) is given below. Inputs are N , C , and $M(=m)$ in the first three lines. This program was used to deduce the behaviors listed in (7), (8), and (9).

```

10  N=128
20  C=.5
30  M=1
40  S1=0
50  S2=8/SQR(N)+16/N+1
60  COM N,C,M
70  F=(N-1)*(1-C)^N
80  PRINT "N=";N,"C=";C,"M=";M,S1;S2
90  S3=(FNS(S1)+FNS(S2))*5
100 S4=0
110 S5=2
120 S6=(S2-S1)*.5
130 S7=2/3
140 FOR S8=1 TO S5-1 STEP 2
150 S4=S4+FNS(S1+S6*S8)
160 NEXT S8
170 Q=(S3+2*S4)*S6*S7+F
180 PRINT USING "M.12DE,7D";Q,S5
190 S3=S3+S4
200 S4=0
210 S5=S5*2
220 S6=S6*.5
230 GOTO 140

```

TR 6327

```
240 DEF FNS(T)
250 COM N,C,M
260 E=EXP(-2*T)
270 D=1-E
280 S=1+E
290 A=(D/S)^2
300 RETURN 3*E*D/S^3*FNP(A)*T^M
310 FNEED
320 DEF FNF21(A,B,C,X)
330 F=F4=1
340 FOR F5=1 TO 100
350 F6=F5-1
360 F4=F4*(A+F6)*(B+F6)*X/((C+F6)*F5)
370 F=F+F4
380 IF ABS(F4)<1E-12*ABS(F) THEN 420
390 NEXT F5
400 DISP "100 TERMS AT ";A;B;C;X
410 PAUSE
420 RETURN F
430 FNEED
440 DEF FNP(X)
450 COM N,C,M
460 B=C*X
470 P=(1-X)^(N-2)/(1-B)^(2*N-1)*FNF21(1-N,1-N,1,B)
480 RETURN P
490 FNEED
```

Appendix C

Program for the Evaluation of (13)

```

10  N=16
20  OUTPUT 0;"N =";N
30  PLOTTER IS "GRAPHICS"
40  GRAPHICS
50  DATA .01,.025,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.975,.99
60  DIM G(1:15)      ! GRID LINES
70  READ G(*)
80  Xm=1
90  Ym=FNInuphi(G(15))
100 SCALE 0,Xm,-Ym,Ym
110 FOR X=0 TO Xm STEP Xm/10
120 MOVE X,Ym
130 DRAW X,-Ym
140 NEXT X
150 FOR I=1 TO 15
160 Y=FNInuphi(G(I))
170 MOVE 0,Y
180 DRAW Xm,Y
190 NEXT I
200 PENUP
210 DATA .01,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.99
220 DIM C(1:13)      ! TRUE COHERENCE VALUES
230 READ C(*)
240 FOR I=3 TO 11
250 C=C(I)
260 T=(1-C)^2
270 Bias=T*(1+2*C/N)/N
280 Sigma=(N-1)*T*(2+C*(1-6*C+13*C^2)/N)/(N*(N+1))
290 Sigma=SQR(Sigma)
300 LINE TYPE 3
310 FOR Td=0 TO Xm STEP Xm/100
320 Y=(Td-C-Bias)/Sigma
330 PLOT Td,Y
340 NEXT Td
350 PENUP
360 LINE TYPE 1
370 FOR Td=0 TO Xm STEP Xm/100
380 Tc=Td
390 Y=FNCdfmsc(N,C,Tc)
400 PLOT Td,FNInuphi(Y)
410 IF Y>.99 THEN 430
420 NEXT Td
430 PENUP
440 NEXT I
450 NEXT K
460 END

```

```

470 DEF FNCdfmsc(N,C,Ct)
480 O=1-Ct
490 P=C*Ct
500 H=1-P
510 R=O/H
520 Tnew=0
530 S=T=1
540 FOR K=1 TO N-2
550 Told=Tnew
560 Tnew=T
570 T=((2*K-1+(N-K)*P)*Tnew-(K-1)*O*Told)*R/K
580 S=S+T
590 NEXT K
600 P=Ct*((1-C)/H)^N*S
610 RETURN P
620 FNEND
630 DEF FNInvphi(X)
640 IF (X>=0) AND (X<=1) THEN 670
650 PRINT "ARGUMENT ";X;" IS DISALLOWED FOR INVERSE PHI FUNCTION"
660 STOP
670 IF (X>0) AND (X<1) THEN 700
680 P=9.999999999999999E99*(2*X-1)
690 GOTO 750
700 P=X
710 IF X>.5 THEN P=.5-(X-.5)
720 P=SQR(-2*LOG(P))
730 P=P-(2.515517+P*(.802853+P*(.010328)))/(1+P*(1.432788+P*(.189269+P*(.001308)))
740 IF X<.5 THEN P=-P
750 RETURN P
760 FNEND
770 DEF FNTanh(X)
780 S=EXP(2*X)
790 RETURN (S-1)/(S+1)
800 FNEND
810 DEF FNArctanh(X)
820 RETURN .5*LOG((1+X)/(1-X))
830 FNEND

```


Appendix D **Program for the Evaluation of (12)**

The functions Cdfmsc, Invphi, Tanh, and Arctanh have already been listed in appendix C.

```

10  N=8
20  OUTPUT 0;"N =";N
30  PLOTTER IS "GRAPHICS"
40  GRAPHICS
50  DATA .01,.025,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.975,.99
60  DIM G(1:15)          ! GRID LINES
70  READ G(*)
80  Xm=4
90  Ym=FNInvphi(G(15))
100 SCALE 0,Xm,-Ym,Ym
110 FOR X=0 TO Xm STEP Xm/8
120 MOVE X,Ym
130 DRAW X,-Ym
140 NEXT X
150 FOR I=1 TO 15
160 Y=FNInvphi(G(I))
170 MOVE 0,Y
180 DRAW Xm,Y
190 NEXT I
200 PENUP
210 DATA .01,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.99
220 DIM C(1:13)          ! TRUE COHERENCE VALUES
230 READ C(*)
240 FOR I=1 TO 13
250 C=C(I)
260 B=(1-C^2)/(2*(N-1))
270 Mean=FNArctanh(SQR(C+B))
280 Sigma=1/SQR(2*(N-1))
290 LINE TYPE 3
300 FOR Td=0 TO Xm STEP Xm/100
310 Y=(Td-Mean)/Sigma
320 PLOT Td,Y
330 NEXT Td
340 PENUP
350 LINE TYPE 1
360 FOR Td=0 TO Xm STEP Xm/100
370 Tc=FHTanh(Td)^2
380 Y=FNcdfmsc(N,C,Tc)
390 PLOT Td,FNInvphi(Y)
400 IF Y>.99 THEN 420
410 NEXT Td
420 PENUP
430 NEXT I
440 NEXT K
450 END

```

Appendix E
Program for the Evaluation of (16)

The functions Invphi and Tanh have already been listed in appendix C.

```
10  INPUT P,N,C          ! PROBABILITY,PIECES,COHERENCE
20  Alpha=SQR(C+(1-C*C)/(2*(N-1)))
30  T=FNInvphi(P)/SQR(2*(N-1))
40  Beta=FNtanh(T)
50  A=((Alpha+Beta)/(1+Alpha*Beta))^2
60  PRINT A,P;N;C
70  END
```

Appendix F **Program for the Evaluation of (23)**

The functions Cdfmsc, Invphi, and Arctanh have already been listed in appendix C.

```

10  C=.1
20  N=32
30  PRINT "C =";C,"N =";N
40  B=(1-C*C)/(2*(N-1))
50  Mean=FNArctanh(SQR(C+B))
60  Sigma=1/SQR(2*(N-1))
70  PLOTTER IS "GRAPHICS"
80  GRAPHICS
90  SCALE 0,1,-.5,3.5
100 GRID .1,.5
110 PENUP
120 LINE TYPE 3
130 FOR X=0 TO .995 STEP .005
140 Y=FNArctanh(SQR(X))          ! ORIGINAL NON-LINEARITY
150 PLOT X,Y
160 NEXT X
170 PENUP
180 LINE TYPE 1
190 FOR X=.002 TO .998 STEP .002
200 P1=FNcdfmsc(N,C,X)
210 IF P1<.01 THEN 250
220 IF P1>1-.01 THEN 260
230 Y=Mean+Sigma*FNInvphi(P1)    ! DESIRED NON-LINEARITY
240 PLOT X,Y
250 NEXT X
260 END

```

Appendix G **Program for Evaluation of (26)**

The functions Cdfmsc, Invphi, Tanh, and Arctanh have already been listed in appendix C.

```

10  N=32
20  OUTPUT 0;"N=";N
30  PLOTTER IS "GRAPHICS"
40  GRAPHICS
50  DATA .01,.025,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.975,.99
60  DIM G(1:15)          ! GRID LINES
70  READ G(*)
80  Xm=4
90  Ym=FNInvphi(G(15))
100 SCALE 0,Xm,-Ym,Ym
110 FOR X=0 TO Xm STEP Xm/8
120 MOVE X,Ym
130 DRAW X,-Ym
140 NEXT X
150 FOR I=1 TO 15
160 Y=FNInvphi(G(I))
170 MOVE 0,Y
180 DRAW Xm,Y
190 NEXT I
200 PENUP
210 DATA .01,.05,.1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.99
220 DIM C(1:13)          ! TRUE COHERENCE VALUES
230 READ C(*)
240 FOR I=1 TO 13
250 C=C(I)
260 B=(1-C^2)/(2*(N-1))
270 Mean=FNArctanh(SQR(C+B))
280 Sigma=1/SQR(2*(N-1))
290 LINE TYPE 3
300 FOR Td=0 TO Xm STEP Xm/100
310 Y=(Td-Mean)/Sigma
320 PLOT Td,Y
330 NEXT Td
340 PENUP
350 LINE TYPE 1
360 FOR Td=0 TO Xm STEP Xm/100
370 Tc=FNtanh(Td)
380 Tc=FNInvg(Tc,N)
390 Y=FNCdfmsc(N,C,Tc)
400 PLOT Td,FNInvphi(Y)
410 IF Y>.99 THEN 430
420 NEXT Td
430 PENUP
440 NEXT I
450 NEXT K
460 END

```

! DISTORTION IS g(N)

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```
470 DEF FNInvG(Y,N)
480 X1=(Y+1.5/N)^2
490 X2=X1+.01
500 F1=SQR(X1)+.5*(1-X1)^(1*N)*LOG(.5*(N-1)*X1)/(N+2)-Y
510 F2=SQR(X2)+.5*(1-X2)^(1*N)*LOG(.5*(N-1)*X2)/(N+2)-Y
520 IF ABS(F1-F2)<1E-10 THEN RETURN X2
530 T=(X1+F2-X2+F1)/(F2-F1)
540 X1=X2
550 X2=T
560 F1=F2
570 GOTO 510
580 FNEND
```

References

- 1 G. C. Carter and A. H. Nuttall, "Evaluation of the Statistics of the Estimate of Magnitude-Squared Coherence," NUSC Technical Memorandum No. TC-193-71, 28 September 1971. Also in *Coherence Estimation*, Scientific and Engineering Studies, NUSC, 1979.
2. P. F. Lee, "An Algorithm for Computing the Cumulative Distribution Function for Magnitude-Squared Coherence Estimates," accepted for publication in *IEEE Transactions on Acoustics, Speech, and Signal Processing*.
3. R. A. Fisher, *Contributions to Mathematical Statistics*, Wiley, NY, 1950. Chapter 14 was originally published as "The General Sampling Distribution of the Multiple Correlation Coefficient," *Proceedings of the Royal Society, Series A*, vol. 121, pp. 654-673, 1928.
4. A. H. Nuttall and G. C. Carter, *Approximations for Statistics of Coherence Estimators*, NUSC Technical Report 5291, 9 March 1976.
5. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, NY, 1965.
6. *Handbook of Mathematical Functions*, U.S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series No. 55, U.S. Government Printing Office, June 1964.
7. N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals*, Holt, Rinehart, and Winston, NY, 1975.

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